VOLUME ELEMENT IN TERMS OF METRIC DETERMINANT

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercises 7-8.

Post date: 16 Apr 2020.

Zee shows with his eqn 23 in Chapter I.5 that the volume element in D-dimensional space is invariant if multiplied by the square root of the determinant of the metric matrix. That is, the quantity

$$d^D x\sqrt{g} \tag{1}$$

is invariant, where

$$g = \det g_{\mu\nu} \tag{2}$$

A simple example of this is in two dimensions using polar coordinates, where the metric is

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{3}$$

The metric is

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \tag{4}$$

with determinant

$$g = r^2 \tag{5}$$

The 'volume' element here is an element of area, and is given by

$$d^2 x\sqrt{g} = r\ dr\ d\theta \tag{6}$$

For the unit circle, an element of length is obtained from 3 with $dr = 0$, so, with $r = 1$

$$ds = r\ d\theta = d\theta \tag{7}$$

Technically, a line element on the unit circle is a one-dimensional space with $ds^2 = d\theta^2$, and $g_{\mu\nu}$ has only one element, namely $g_{\theta\theta} = 1$. Thus the determinant is $g = 1$ and the formula 1 holds with $D = 1$.

Now in 3 dimensions, we consider the metric

$$ds^2 = a^2 dx^2 + b^2 dy^2 + c^2 dz^2 \tag{8}$$

so that

$$g_{\mu\nu} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{bmatrix} \tag{9}$$

$$g = \det g_{\mu\nu} = (abc)^2 \tag{10}$$

In this case, a volume element consists of

$$(a\ dx)(b\ dy)(c\ dz) = abc\ dxdydz = \sqrt{g} d^3x \tag{11}$$