UNIT SPHERES IN HIGHER DIMENSIONS

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercises 9-10.

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The unit circle (embedded in two dimensional Euclidean space) and unit sphere (embedded in three dimensions) can be generalized to a d-dimensional 'surface' embedded in a $d+1$ dimensional Euclidean space by defining the Pythagorean relation

$$P_d \equiv \left(X^1\right)^2 + \left(X^2\right)^2 + \ldots + \left(X^{d+1}\right)^2 = 1 \tag{1}$$

where

$$X^1 = \cos\theta_1 \tag{2}$$
$$X^2 = \sin\theta_1 \cos\theta_2 \tag{3}$$
$$\vdots \tag{4}$$
$$X^d = \sin\theta_1 \ldots \sin\theta_{d-1} \cos\theta_d \tag{5}$$
$$X^{d+1} = \sin\theta_1 \ldots \sin\theta_{d-1} \sin\theta_d \tag{6}$$

where $0 \le \theta_i \le \pi$ for $1 \le i \le d-1$ and $0 \le \theta_d \le 2\pi$. If you want to relate this to the usual 3-d spherical coordinates on a unit sphere, note that $X^1 = z$, $X^2 = x$ and $X^3 = y$, and $\theta_1 = \theta$ and $\theta_2 = \phi$.

We first need to verify that 1 holds for all higher dimensions. We know that it holds for $d=1$ (unit circle) and $d=2$ (unit sphere), so it seems reasonable to use induction to do the proof for higher dimensions. We assume 1 is true for some value d and try to prove from this that it is therefore also true for $d+1$.

We'll use the shorthand notation to save typing:

$$s_i \equiv \sin\theta_i \tag{7}$$
$$c_i \equiv \cos\theta_i \tag{8}$$

To go from P_d to P_{d+1} in 1, we have

$$P_{d+1} = P_d - (s_1 s_2 \ldots s_{d-1} s_d)^2 + (s_1 s_2 \ldots s_d c_{d+1})^2 + (s_1 s_2 \ldots s_d s_{d+1})^2 \tag{9}$$

$$= P_d + (s_1 s_2 \ldots s_{d-1} s_d)^2 \left(-1 + c_{d+1}^2 + s_{d+1}^2\right) \tag{10}$$

$$= P_d \tag{11}$$

where to get the last line, we used

$$c_{d+1}^2 + s_{d+1}^2 = \cos^2\theta_{d+1} + \sin^2\theta_{d+1} = 1 \tag{12}$$

Therefore $P_d = 1$ for all d, which means 1 is always true.

We would now like to prove the metric formula given in Zee's question, namely

$$ds_d^2 = \sum_{i=1}^{d+1} \left(dX^i\right)^2 \tag{13}$$

$$= d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \ldots + \sin^2\theta_1 \ldots \sin^2\theta_{d-1} d\theta_d^2 \tag{14}$$

Again, we can verify that the formula is true for $d = 1$ and $d = 2$, so it looks like induction is a good method to try here too. However, the calculations get a lot messier since, in order to calculate differentials, we need to use the product rule, so each term in the differential dX^i will expand into numerous other terms. We'll again use the shorthand notation 7, with the additional definitions

$$s_{ss} \equiv s_1 s_2 \ldots s_{d-1} \tag{15}$$

$$ds_{ss} \equiv d(s_1 s_2 \ldots s_{d-1}) \tag{16}$$

$$d_i \equiv d\theta_i \tag{17}$$

We start by assuming that 14 is true for some value d and try to prove that it's true for $d+1$. We have, using the chain and product rules repeatedly:

$$ds_{d+1}^2 = ds_d^2 - [d(s_{ss} s_d)]^2 + [d(s_{ss} s_d c_{d+1})]^2 + [d(s_{ss} s_d s_{d+1})]^2 \tag{18}$$

$$= ds_d^2 - [ds_{ss} s_d + s_{ss} d_d]^2 + \tag{19}$$

$$[ds_{ss} s_d c_{d+1} + s_{ss} c_d d_d c_{d+1} - s_{ss} s_d s_{d+1} d_{d+1}]^2 + \tag{20}$$

$$[ds_{ss} s_d s_{d+1} + s_{ss} c_d d_d s_{d+1} + s_{ss} s_d c_{d+1} d_{d+1}]^2 \tag{21}$$

As you can see, the algebra gets very messy when all the squares are multiplied out, so I used Maple to do the algebra and trigonometric simplifications. The result is that

$$ds_{d+1}^2 = ds_d^2 + d_{d+1}^2 s_{ss}^2 \sin^2\theta_d \tag{22}$$

$$= ds_d^2 + \sin^2\theta_1 \ldots \sin^2\theta_{d-1} \sin^2\theta_d d\theta_{d+1}^2 \tag{23}$$

which agrees with 14 with d replaced by $d+1$.

Finally, we can show an iteration formula. Looking at 14, we see that we can generate ds_{d+1}^2 from ds_d^2 by relabelling all the angles to be one index higher, multplying the result by $\sin^2\theta_1$ and then adding $d\theta_1^2$ to restore the first term. That is, if we first relabel all the angles, we get

$$d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \ldots + \sin^2\theta_1 \ldots \sin^2\theta_{d-1} d\theta_d^2 \rightarrow$$
$$d\theta_2^2 + \sin^2\theta_2 d\theta_3^2 + \ldots + \sin^2\theta_2 \ldots \sin^2\theta_d d\theta_{d+1}^2 \tag{24}$$

We next multiply by $\sin^2\theta_1$ to get

$$\sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2 + \ldots + \sin^2\theta_1 \sin^2\theta_2 \ldots \sin^2\theta_d d\theta_{d+1}^2 \tag{25}$$

We then add on $d\theta_1^2$ to get ds_{d+1}^2:

$$ds_{d+1}^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \sin^2\theta_1 \sin^2\theta_2 d\theta_3^2 + \ldots + \sin^2\theta_1 \sin^2\theta_2 \ldots \sin^2\theta_d d\theta_{d+1}^2 \tag{26}$$

If we label the angles in ds_{d-1}^2 from 2 to d rather than from 1 to $d-1$ (it's just a label, after all) we then get

$$ds_d^2 = d\theta_1^2 + \sin^2\theta_1 ds_{d-1}^2 \tag{27}$$

PINGBACKS

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