AREA OF UNIT SPHERES IN HIGHER DIMENSIONS

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercise 11.

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We now consider a general formula for calculating the area of a unit sphere S^d of d dimensions embedded in Euclidean space E^{d+1} of one higher dimension than the sphere. Familiar examples are S^1 (a circle) embedded in E^2 (a plane), and S^2 (the usual sphere) embedded in 3-d Euclidean space E^3.

The general formula is an integral over the d angle variables that define a point on the surface of S^d. If we call the generalized area A^d, then

$$A^d = \int_0^{2\pi} d\theta_d \int_0^{\pi} d\theta_1 \ldots \int_0^{\pi} d\theta_{d-1}\sqrt{g} \tag{1}$$

where g is the determinant of the metric $g_{\mu\nu}$. Note that the last angle θ_d runs from 0 to 2π while all the other angles run from 0 to π. For example, with $d=1$, the circle has only one angle variable which runs from 0 to 2π. With an ordinary sphere S^2, the polar angle (usually written as $\theta_1=\theta$) runs from 0 to π while the azimuthal angle $\theta_2=\phi$ runs from 0 to 2π.

We can get the metric tensor from the general formula for ds_d^2.

$$ds_d^2 = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \ldots + \sin^2\theta_1 \ldots \sin^2\theta_{d-1} d\theta_d^2 \tag{2}$$

From this we see that $g_{\mu\nu}$ is diagonal, with diagonal entry given by $g_{11}=0$ and, for $i>1$

$$g_{ii} = \prod_{j=1}^{i-1} \sin^2\theta_j \tag{3}$$

Zee asks us to check the answer for $d=2$ and $d=3$ but I think he means $d=1$ and $d=2$ to get the circle and ordinary sphere, as described above. For $d=1$ we have g_{11} =1 so the metric matrix is just a 1×1 matrix with determinant $g=1$, and

$$A^1 = \int_0^{2\pi} d\theta_1 = 2\pi \tag{4}$$

which is the circumference of a unit circle.

For $d = 2$, we have

$$A^2 = \int_0^{2\pi} d\theta_2 \int_0^{\pi} d\theta_1 \sqrt{g} \tag{5}$$

In this case,

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta_1 \end{bmatrix} \tag{6}$$

so $g = \sin^2\theta_1$ and we have

$$A^2 = \int_0^{2\pi} d\theta_2 \int_0^{\pi} d\theta_1 \sin\theta_1 = 4\pi \tag{7}$$

which is the area of a unit sphere embedded in 3-d Euclidean space.

For $d = 3$, we have, using 2

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2\theta_1 & 0 \\ 0 & 0 & \sin^2\theta_1 \sin^2\theta_2 \end{bmatrix} \tag{8}$$

$$\sqrt{g} = \sin^2\theta_1 \sin\theta_2 \tag{9}$$

The area is then

$$A^3 = \int_0^{2\pi} d\theta_3 \int_0^{\pi} d\theta_2 \sin\theta_2 \int_0^{\pi} d\theta_1 \sin^2\theta_1 \tag{10}$$

$$= 2\pi^2 \tag{11}$$

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