

SQUASHED SPHERE

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercise 12.

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We consider a 'squashed sphere' with the metric

$$ds^2 = (b^2 + a^2 \cos^2 \theta) d\theta^2 + \frac{(b^2 + a^2)^2}{b^2 + a^2 \cos^2 \theta} \sin^2 \theta d\phi^2 \quad (1)$$

where a and b are constants, and the angle variables have the usual ranges: $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$.

We can find the length of the equator and of a circle of fixed longitude going through the two poles. For the equator $\theta = \frac{\pi}{2}$ and $d\theta = 0$, so the length element is

$$ds = \frac{b^2 + a^2}{b} d\phi \quad (2)$$

so the circumference C_E of the equator is

$$C_E = \int_0^{2\pi} \frac{b^2 + a^2}{b} d\phi = \frac{2\pi}{b} (b^2 + a^2) \quad (3)$$

For a polar circle, we can take $\phi = 0$, $d\phi = 0$ and let θ vary from 0 to π , which gives us a semicircle, so we double this result to get the full circle. The polar circumference is then C_P

$$C_P = 2 \int_0^\pi d\theta \sqrt{b^2 + a^2 \cos^2 \theta} \quad (4)$$

Running this integral through Maple, we find that it is a complete elliptic integral, denoted by `EllipticE` with a single argument. If we take Zee's hint to set $b = a$, then we get

$$C_P = 2a \times 2\sqrt{2} \text{EllipticE} \left(\frac{\sqrt{2}}{2} \right) \quad (5)$$

This has a numerical value of approximately

$$C_P \approx 7.64a \quad (6)$$

To calculate the area, we use the formula

$$A = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sqrt{g} \quad (7)$$

where the metric tensor can be read off from 1

$$g_{\mu\nu} = \begin{bmatrix} b^2 + a^2 \cos^2 \theta & 0 \\ 0 & \frac{(b^2 + a^2)^2}{b^2 + a^2 \cos^2 \theta} \sin^2 \theta \end{bmatrix} \quad (8)$$

The determinant is

$$g = (b^2 + a^2)^2 \sin^2 \theta \quad (9)$$

so we have for the area

$$A = (b^2 + a^2) \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \quad (10)$$

$$= 4\pi (b^2 + a^2) \quad (11)$$