CONFORMALLY FLAT SPACES

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercise 14.

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A space with a metric

$$ds^2 = \Omega^2(x)\, ds^2_{\text{flat}} \tag{1}$$

is defined to be *conformally flat*. That is, the conformally flat metric is created by multiplying all components of the flat metric by the same function, although the function itself can vary over the space (hence the dependence on x in $\Omega(x)$).

The length between two points in the conformally flat space is not necessarily the same as in the flat space, since the function Ω effectively scales all lengths. Angles, however, are preserved. We can see this by using the scalar or dot product (generalized to higher dimensions, if required). We can write the scalar product between two vectors u^μ and v^ν as

$$\boldsymbol{u}\cdot\boldsymbol{v} = g_{\mu\nu}u^\mu v^\nu \tag{2}$$

where $g_{\mu\nu}$ is the metric tensor. The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined by

$$\cos\theta = \frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}|\,|\boldsymbol{v}|} \tag{3}$$

$$= \frac{g_{\mu\nu}u^\mu v^\nu}{\sqrt{g_{\mu\nu}u^\mu u^\nu}\sqrt{g_{\mu\nu}v^\mu v^\nu}} \tag{4}$$

We define the metric in the flat space to be $g_{\mu\nu}^{\text{flat}}$ and the metric in the conformally flat space to be just $g_{\mu\nu}$. Thus in the flat space, we have

$$\cos\theta_{\text{flat}} = \frac{g_{\mu\nu}^{\text{flat}}u^\mu v^\nu}{\sqrt{g_{\mu\nu}^{\text{flat}}u^\mu u^\nu}\sqrt{g_{\mu\nu}^{\text{flat}}v^\mu v^\nu}} \tag{5}$$

Since the transformation 1 scales all components of the flat metric by the same amount at any given point x, the angle in the transformed space is

$$\cos\theta = \frac{\Omega^2(x)\, g_{\mu\nu}^{\text{flat}} u^\mu v^\nu}{\sqrt{\Omega^2(x)\, g_{\mu\nu}^{\text{flat}} u^\mu u^\nu}\sqrt{\Omega^2(x)\, g_{\mu\nu}^{\text{flat}} v^\mu v^\nu}} \tag{6}$$

$$= \frac{g_{\mu\nu}^{\text{flat}} u^\mu v^\nu}{\sqrt{g_{\mu\nu}^{\text{flat}} u^\mu u^\nu}\sqrt{g_{\mu\nu}^{\text{flat}} v^\mu v^\nu}} \tag{7}$$

$$= \cos\theta_{\text{flat}} \tag{8}$$

The scaling factor $\Omega(x)$ cancels out in the formula for the angle.

An example of a conformally flat space is provided by our earlier example of the stereographic projection of the sphere. The metric of the sphere in terms of the projected coordinates is

$$ds^2 = \frac{1}{\left(1+\frac{\rho^2}{4L^2}\right)^2}\left(d\rho^2 + \rho^2 d\phi^2\right) \tag{9}$$

where L is the radius of the sphere and ρ is the radius of the projected coordinate on the plane. Since $d\rho^2 + \rho^2 d\phi^2$ is the usual metric in polar coordinates of the plane, it represents a flat space. Thus the sphere's metric can be written as a scaling factor

$$\Omega(\rho) = \frac{1}{\left(1+\frac{\rho^2}{4L^2}\right)^2} \tag{10}$$

multiplied by the metric for a flat space. This also illustrated the fact that a conformally flat space is not necessarily a true flat space, since the sphere is obviously a curved space, but is still conformally flat since it satisfies 1.