TORUS METRIC

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.5, Exercise 16.

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To find the metric on the surface of a torus, we can first consider what coordinates we need to specify a position on a torus. See Figure 1. The top view shows the torus lying flat on the xy plane. The radius of the torus (from its centre to halfway through the tube) is L and the angle measured counterclockwise from the x axis is ϕ, so $0 \le \phi < 2\pi$.

The inset figure at the bottom right shows a vertical cross-section through the tube of the torus. The radius of the tube is a and the angle measured counterclockwise from the top of the torus is θ, so $0 \le \theta < 2\pi$. Thus the radius of the torus from its centre to the inner ring is $L-a$ and to the outer ring is $L+a$.

The z (vertical) coordinate of a point on the torus is therefore determined entirely by the vertical cross section, and is

$$z = a\cos\theta \tag{1}$$

so that $z=0$ in the horizontal plane that bisects the torus horizontally (the xy plane), where $\theta = \frac{\pi}{2}$.

The radial distance from the centre to a point on the torus is

$$r = L - a\sin\theta \tag{2}$$

My answer differs from Zee's because he uses θ that starts from the bottom of the tube rather than the top so I get a factor of $(L-a\sin\theta)$. I was too lazy to go back and change everything.

so the x and y coordinates are

$$x = r\cos\phi = (L-a\sin\theta)\cos\phi \tag{3}$$

$$y = r\sin\phi = (L-a\sin\theta)\sin\phi \tag{4}$$

The infinitesimal increments are then

$$dx = -a\cos\theta\cos\phi d\theta - (L-a\sin\theta)\sin\phi d\phi \tag{5}$$

$$dy = -a\cos\theta\sin\phi d\theta + (L-a\sin\theta)\cos\phi d\phi \tag{6}$$

$$dz = -a\sin\theta d\theta \tag{7}$$

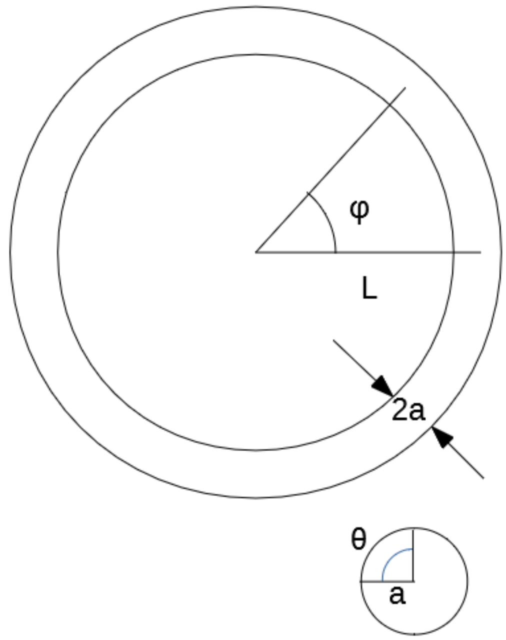

FIGURE 1. Torus. Horizontal view (top) and cross section of tube (bottom).

Squaring and adding gives (using Maple to do the algebra and trig simplifications)

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 && (8)\\ &= a^2 d\theta^2 + \left(L^2 + a^2 - a^2\cos^2\theta - 2La\sin\theta\right) d\phi^2 && (9)\\ &= a^2 d\theta^2 + (L - a\sin\theta)^2 d\phi^2 && (10)\end{aligned}$$

Thus the metric is

$$g_{\mu\nu} = \begin{bmatrix} a^2 & 0 \\ 0 & (L - a\sin\theta)^2 \end{bmatrix} \tag{11}$$

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