ESKIMO MITE METRIC REVISITED

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.6, Exercise 1.

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I should start this post with a disclaimer that I haven't actually solved this problem. I've only sketched out a few ideas.

In Chapter I.6, Zee shows how at some point on a general curved surface, the metric can be approximated by a quadratic form. In Cartesian coordinates, this means that the surface can be approximated by an equation of the form

$$z(x,y) = \frac{1}{2}ax^2 + cxy + \frac{1}{2}by^2 \tag{1}$$

where a, b and c are constants, and we take the surface to have a value of $z(0,0) = 0$. Given this, we have

$$dz = (ax+cy)\,dx + (cx+by)\,dy \tag{2}$$

and the length element is

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{3}$$

$$= \left[1+(ax+cy)^2\right]dx^2 + \left[1+(cx+by)^2\right]dy + (ax+cy)(cx+by)\,dx\,dy \tag{4}$$

The problem asks us to find the relating the coordinates used by the Eskimo mites of problem I.5.2 to the coordinates used here. The Eskimo mites had a metric given by

$$ds^2 = \left(1-\frac{y^2}{3}\right)dx^2 + \left(1-\frac{x^2}{3}\right)dy^2 + \frac{2}{3}xy\,dxdy + \ldots \tag{5}$$

Comparing this to 4 we see an immediate problem: the metric tensor components from 4 are

$$g_{\mu\nu} = \begin{bmatrix} 1+(ax+cy)^2 & \frac{1}{2}(ax+cy)(cx+by) \\ \frac{1}{2}(ax+cy)(cx+by) & 1+(cx+by)^2 \end{bmatrix} \tag{6}$$

I think g_{xy} should be half the value given by Zee, since it occurs twice in the sum $g_{\mu\nu}x^\mu x^\nu$.

The metric used by the mites is

$$g_{\mu\nu}=\left[\begin{array}{cc}\left(1-\frac{y^{2}}{3}\right) & \frac{1}{3}xy\\ \frac{1}{3}xy & \left(1-\frac{x^{3}}{3}\right)\end{array}\right] \tag{7}$$

Comparing the two we see that we'd need to satisfy the conditions

$$(ax+cy)^{2}=-\frac{y^{2}}{3} \tag{8}$$

$$(cx+by)^{2}=-\frac{x^{2}}{3} \tag{9}$$

Clearly these conditions are impossible if we require a, b and c to be real numbers. We can try a more general approach and propose that the surface used by the mites is given by some general function

$$z=f(x,y) \tag{10}$$

where f isn't necessarily of the form 1. In that case, we have (using a subscript to denote a partial derivative as in $f_{x}\equiv\frac{\partial f}{\partial x}$):

$$dz=f_{x}dx+f_{y}dy \tag{11}$$

$$ds^{2}=\left(1+f_{x}^{2}\right)dx^{2}+\left(1+f_{y}^{2}\right)dy^{2}+2f_{x}f_{y}dx\,dy \tag{12}$$

We see that this doesn't help, since for a real function f, f_{x}^{2} and f_{y}^{2} are again both positive. Even allowing f to be complex doesn't solve the problem. We would need to solve the three equations

$$f_{x}^{2}=-\frac{y^{2}}{3} \tag{13}$$

$$f_{y}^{2}=-\frac{x^{2}}{3} \tag{14}$$

$$2f_{x}f_{y}=\frac{2}{3}xy \tag{15}$$

The first two equations give us

$$f_{x}=\pm\frac{y}{\sqrt{3}}i \tag{16}$$

$$f_{y}=\pm\frac{x}{\sqrt{3}}i \tag{17}$$

We can integrate f_{x} to get

$$f(x,y) = \pm\frac{xy}{\sqrt{3}}i + g(y) \tag{18}$$

where $g(y)$ is some function of y alone.

Integrating f_y we get

$$f(x,y) = \pm\frac{xy}{\sqrt{3}}i + h(x) \tag{19}$$

and $h(x)$ is some function of x alone. Equating the two shows us that $g = h = 0$, and we must choose the *same* sign for the other term. However, this makes the third condition 15 impossible to satisfy, since we then get $2f_x f_y = -\frac{2}{3}xy$.

The original Eskimo mite metric was derived using the approximation to the spherical metric of the form

$$x = \theta\cos\phi \tag{20}$$

$$y = \theta\sin\phi \tag{21}$$

$$ds^2 = d\theta^2 + \left(\theta - \frac{\theta^3}{3!}\right)^2 d\phi^2 \tag{22}$$

That is, we approximated $\sin\theta$ by θ in defining x and y, but then approximated $\sin^2\theta$ in ds^2 by $\left(\theta - \frac{\theta^3}{3!}\right)^2$ and kept terms up to second order. This incosistency may be the cause of the problem, but I have to confess I can't see how to fix it. Comments welcome.