CURVATURE OF A TORUS

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.6, Exercise 2.

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We wish to find the curvature of a torus using the tangent plane method described in Chapter I.6. We consider a torus with a radius (to the midpoint of the ring) of L and a ring radius of a. See Fig. 1 The angle ϕ measures the angular distance around the ring measured from the x axis, and θ measures the angular distance inside the ring measured from the top of the ring.

On one level, the answer to this problem can be read off from the torus's definition. We consider a plane tangent to the inner surface of the ring (that is, at radius $L-a$) at the pont where the torus crosses the x axis. Then we want the plane tangent to the torus at the point $(L-a,0,0)$. This is a vertical plane. The torus has a curvature of $-(L-a)$ in the y direction (negative because the circle curves backwards). The curvature in the z direction is the curvature of the circle that is the cross-section of the tube, which is a

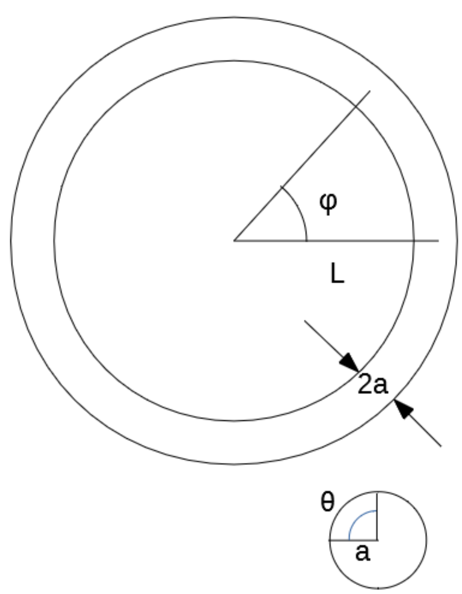

FIGURE 1. A torus.

(positive because it curves outwards) Thus the curvature at this point has opposite signs in the y and z directions, since the point $(L-a,0,0)$ is a saddle point.

On the outer surface at point $(L+a,0,0)$ the curvatures are $-(L+a)$ in the y direction and $-a$ in the z direction. Both curvatures are in the same direction here.

On the top of the torus, the tangent plane is horizontal, and there's a curvature of a in the x direction and 0 in the y direction.

To do this formally using the tangent plane method Zee describes in the book, we start with the equations for the torus:

$$x = r\cos\phi = (L-a\sin\theta)\cos\phi \tag{1}$$
$$y = r\sin\phi = (L-a\sin\theta)\sin\phi \tag{2}$$
$$z = a\cos\theta \tag{3}$$

If we take the point $(L-a,0,0)$ for the tangent plane then, since the plane is vertical and we're interested in curvatures in the y and z directions, we need to consider slight changes in these two variables. That is, we look at

$$\theta = \frac{\pi}{2}+\epsilon \tag{4}$$
$$\phi = \eta \tag{5}$$

where ϵ and η are infinitesimal quantities. This gives us

$$\sin\theta = \sin\left(\frac{\pi}{2}+\epsilon\right) \tag{6}$$
$$= \cos\epsilon \tag{7}$$
$$\approx 1-\frac{\epsilon^2}{2} \tag{8}$$
$$\cos\theta = \cos\left(\frac{\pi}{2}+\epsilon\right) \tag{9}$$
$$= -\sin\epsilon \tag{10}$$
$$\approx -\epsilon \tag{11}$$
$$\cos\eta \approx 1-\frac{\eta^2}{2} \tag{12}$$
$$\sin\eta \approx \eta \tag{13}$$

Plugging these into the coordinate equations 2 we get, keeping up to second order in the infinitesimals:

$$x = \left(L - a\left(1 - \frac{\epsilon^2}{2}\right)\right)\left(1 - \frac{\eta^2}{2}\right) \tag{14}$$

$$\approx L - a + a\frac{\epsilon^2}{2} - (L-a)\frac{\eta^2}{2} \tag{15}$$

$$y = \left(L - a\left(1 - \frac{\epsilon^2}{2}\right)\right)\eta \tag{16}$$

$$\approx (L-a)\eta \tag{17}$$

$$z = -a\epsilon \tag{18}$$

Combining these, we get

$$x = \left(L - a + \frac{z^2}{2a}\right)\left(1 - \frac{\eta^2}{2}\right) \tag{19}$$

$$= L - a + \frac{z^2}{2a} - \frac{y^2}{2(L-a)} \tag{20}$$

Since the curvature R of a quadratic term is given by a $2R$ factor in the denominator, we see that the curvature in the z direction is $+a$ and in the y direction it is $-(L-a)$. We can do similar calculations to get the other two cases mentioned above, but the calculations are essentially the same so I won't repeat them here.