LOCALLY FLAT COORDINATES ON THE SPHERE

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.6, Exercise 3.

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Using a pair of coordinates denoted (κ,ζ), we are given a metric which is, up to first order in the coordinates

$$g_{\kappa\kappa} = 1 \tag{1}$$
$$g_{\zeta\zeta} = 1+2\kappa \tag{2}$$
$$g_{\kappa\zeta} = 0 \tag{3}$$

If we make the transformation

$$\kappa = \omega + \frac{1}{2}\phi^2 \tag{4}$$
$$\zeta = \phi - \omega\phi \tag{5}$$

we can get a metric with no linear terms. We need to use the metric transformation formula given as Zee's equation (8) in chapter I.6:

$$g'_{\lambda\sigma}\left(x'\right) = g_{\mu\nu}\left(x\right)\frac{\partial x^{\mu}}{\partial x'^{\lambda}}\frac{\partial x^{\nu}}{\partial x'^{\sigma}} \tag{6}$$

We therefore need the derivatives:

$$\frac{\partial\kappa}{\partial\omega} = 1 \tag{7}$$
$$\frac{\partial\kappa}{\partial\phi} = \phi \tag{8}$$
$$\frac{\partial\zeta}{\partial\omega} = -\phi \tag{9}$$
$$\frac{\partial\zeta}{\partial\phi} = 1-\omega \tag{10}$$

We then have

$$g'_{\omega\omega} = 1 + (1+2\kappa)\,\phi^2 \tag{11}$$
$$= 1 + \phi^4 + (2\omega+1)\,\phi^2 \tag{12}$$
$$g'_{\omega\phi} = \phi + (1+2\kappa)\,\phi\,(\omega-1) \tag{13}$$
$$= 2\phi\omega^2 + \left(\phi^3-\phi\right)\omega - \phi^3 \tag{14}$$
$$g'_{\phi\phi} = \phi^2 + (1+2\kappa)\,(1-\omega)^2 \tag{15}$$
$$= 1 + 2\omega^3 + \left(\phi^2-3\right)\omega^2 - 2\omega\phi^2 + 2\phi^2 \tag{16}$$

Although the new metric isn't pretty, we can see that there are no terms linear in either ω or ϕ.

We can now relate this to the flat metric in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{17}$$

We assume that we start at some point $P = (r,\theta) = (r_*,0)$. Then if we move a small distance away from this point and take (κ,ζ) to be the deviation we have

$$\kappa = r - r_* \tag{18}$$
$$\zeta = \theta \tag{19}$$

where we're assuming that r is close to r_* and θ is small. Then we have

$$ds^2 = dr^2 + (\kappa + r_*)^2\,d\theta^2 \tag{20}$$
$$= d\kappa^2 + \left(r_*^2 + 2\kappa r_* + \kappa^2\right)d\zeta^2 \tag{21}$$
$$= d\kappa^2 + \left(1 + \frac{2\kappa}{r_*} + \frac{\kappa^2}{r_*^2}\right)(r_* d\zeta)^2 \tag{22}$$
$$= r_*^2\left[\frac{d\kappa^2}{r_*^2} + \left(1 + \frac{2\kappa}{r_*} + \frac{\kappa^2}{r_*^2}\right)d\zeta^2\right] \tag{23}$$

If we scale $\kappa \to \frac{\kappa}{r_*}$ and $\zeta \to r_*\zeta$ this is equivalent to

$$ds^2 = d\kappa^2 + (\kappa+1)^2\,d\zeta^2 \tag{24}$$

Thus the metric is the same as the flat metric in polar coordinates with $\kappa+1$ playing the role of r and ζ playing the role of θ. Since we're taking κ to be very small, we can assume that $\kappa \ll 1$. Also, we're assuming θ is the deviation from 0 angle.