LOCALLY FLAT COORDINATES IN THE POINCARÉ HALF-PLANE

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.6, Exercise 4.

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The Poincaré half-plane is defined for $y>0$ by the metric

$$ds^2=\frac{1}{y^2}\left(dx^2+dy^2\right) \tag{1}$$

The problem is to find a pair of coordinates (u,v) that give a locally flat metric around, say, the point

$$P=(x,y)=(0,y_*) \tag{2}$$

Note that we can't take $y=0$ since the metric blows up at that value.

A locally flat metric is one where

$$ds^2=du^2+dv^2 \tag{3}$$

at some particular point. Thus it would seem that the original metric is locally flat for $y=1$.

In general, we're looking for (u,v) that satisfies

$$\begin{aligned} du&=\frac{dx}{y}\\ dv&=\frac{dy}{y} \end{aligned} \tag{4}$$

We can integrate these equations to give us

$$\begin{aligned} u&=\frac{x}{y}\\ v&=\ln\frac{y}{y_*} \end{aligned} \tag{5}$$

These solutions have the values $(u,v)=(0,0)$ at the point P.

The solution given at the back of the book defines (u,v) implicitly by stating

$$x = y_*(u + uv + \ldots) \tag{6}$$

$$y = y_*\left(v + \frac{1}{2}\left(v^2 - u^2\right) + \ldots\right) \tag{7}$$

It's something of a mystery as to how these definitions could give us the values of u and v for a locally flat metric. The best explanation I've found is this post on the Physics Forums site. First of all, it seems that Zee's solution contains a typo, in that the y equation should be

$$y = y_*\left(1 + v + \frac{1}{2}\left(v^2 - u^2\right) + \ldots\right) \tag{8}$$

since the original equation gives $y = 0$ for $u = v = 0$, which as we've seen, is not an admissible value since the metric blows up there. Using 8 and 6 give the point P for $u = v = 0$ as required. We then get

$$dx = y_*(du + u\,dv + v\,du) \tag{9}$$

$$dy = y_*(dv + v\,dv - u\,du) \tag{10}$$

This gives for the metric (after cancelling terms):

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right) \tag{11}$$

$$= \frac{y_*^2}{y^2}\left(u^2 + (1+v)^2\right)\left(du^2 + dv^2\right) \tag{12}$$

$$= \frac{u^2 + (1+v)^2}{(1+v)^2 + \frac{1}{2}\left(v^2 - u^2\right)}\left(du^2 + dv^2\right) \tag{13}$$

where the last line keeps only up to second order terms in y from 8. At the point P, $y = y_*$ and $u = v = 0$ so this does indeed give us $ds^2 = du^2 + dv^2$. This solution is valid for any value of $y_* > 0$.

For a locally flat set of coordinates, we'd also like the metric tensor to have no first order terms in its expansion about P. From 13, the metric in the (u, v) coordinates is

$$g_{uu} = g_{vv} = \frac{u^2 + (1+v)^2}{(1+v)^2 + \frac{1}{2}\left(v^2 - u^2\right)} \tag{14}$$

$$g_{uv} = g_{vu} = 0 \tag{15}$$

Using Maple to calculate the first derivatives and simplify the result, we have

$$\frac{\partial g_{uu}}{\partial u} = \frac{4u\left(4v^2+6v+3\right)}{\left(u^2-3v^2-4v-2\right)^2} \tag{16}$$

$$\frac{\partial g_{vv}}{\partial u} = -\frac{16u^2v+12u^2+4v^2+4v}{\left(u^2-3v^2-4v-2\right)^2} \tag{17}$$

Both these derivatives are zero at $u=v=0$ so the metric has no linear terms.

The question remains as to how we could come up with Zee's answer starting from 5. I think the key is to recognize that 5 isn't actually a complete solution, since we were a bit sloppy in the integration. Going back to 4, we need to see that x could have an extra function of v alone, since then calculating $\frac{\partial x}{\partial u}$ would eliminate this extra function and it wouldn't be visible in the expression for du. Similarly y could have an extra function of u alone, for the same reason. Thus if we solve 5 for x and y we would get

$$y = y_* e^y g(u) \tag{18}$$

$$x = yu + f(v) \tag{19}$$

$$= y_* u e^v g(u) + f(v) \tag{20}$$

where f and g are functions to be determined. If we expand the exponential, we get

$$x = y_* u\left(1+v+\frac{v^2}{2!}+\ldots\right) g(u) + f(v) \tag{21}$$

$$y = y_*\left(1+v+\frac{v^2}{2!}+\ldots\right) g(u) \tag{22}$$

We would then need to calculate dx^2+dy^2 and try to find functions f and g that give a locally flat metric around P. If we save terms only up to second order in u and v (since we're expanding around $u=v=0$, so we are taking u and v to be small), we could then presumably find that taking

$$\begin{aligned} f(v) &= 0 \\ g(u) &= 1-\frac{u^2}{2} \end{aligned} \tag{23}$$

would do the trick.

All of this is a bit hand-wavy, and I would think Zee could make this section a lot clearer if he gave a couple of examples of calculating locally flat coordinates in the text.