CURVATURE OF A 2-DIMENSIONAL SURFACE

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References: Anthony Zee, *Einstein Gravity in a Nutshell*, (Princeton University Press, 2013) - Chapter I.6, Exercise 5.

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In chapter I.6, Zee shows that by a suitable coordinate transformation, the metric can be transformed into a form that has no linear dependence on the coordinates. That is, we can write the metric as

$$g_{\mu\nu}=\delta_{\mu\nu}+B_{\mu\nu,\lambda\sigma}x^{\lambda}x^{\sigma}+\ldots \tag{1}$$

where the $B_{\mu\nu,\lambda\sigma}$ are coefficients to be determined.

In exercise I.6.5, we are asked to show that for a 2-dimensional surface (such as a sphere or paraboloid), a certain combination of the $B_{\mu\nu,\lambda\sigma}$ coefficients gives the curvature of the surface at the origin (he doesn't explicitly say it's at the origin, but the whole argument relies on choosing an origin and expanding $g_{\mu\nu}$ about it, so I'm assuming that's what he means.

I'm not sure this is the most general solution, but if we assume that the surface can be approximated at the origin by a paraboloid of the form given on Zee's page 83:

$$z=\frac{1}{2}ax^{2}+cxy+\frac{1}{2}by^{2} \tag{2}$$

then he shows that the metric is

$$\begin{aligned}
g_{xx}&=1+(ax+cy)^{2}\\
g_{yy}&=1+(by+cx)^{2}\\
g_{xy}&=(ax+cy)(by+cx)
\end{aligned} \tag{3}$$

This relies on the usual expression for the metric:

$$ds^{2}=g_{xx}dx^{2}+2g_{xy}dx\;dy+g_{yy}dy^{2} \tag{4}$$

For the dimension $D=2$, we can expand 1 (using $x=x^{1}$ and $y=x^{2}$) to get

$$\begin{aligned} g_{xx} &= 1 + B_{11,11}x^2 + 2B_{11,12}xy + B_{11,22}y^2 \\ g_{yy} &= 1 + B_{22,11}x^2 + 2B_{22,12}xy + B_{22,22}y^2 \\ g_{xy} &= B_{12,11}x^2 + 2B_{12,12}xy + B_{12,22}y^2 \end{aligned} \tag{5}$$

Expanding 3 we have

$$\begin{aligned} g_{xx} &= 1 + a^2x^2 + 2acxy + c^2y^2 \\ g_{yy} &= 1 + b^2y^2 + 2bcxy + c^2x^2 \\ g_{xy} &= acx^2 + \left(ab + c^2\right)xy + bcy^2 \end{aligned} \tag{6}$$

Since x and y are arbitrary (although small), we can equate coefficients in 5 and 6 to get (I'm considering only the coefficients suggested by Zee in the problem statement):

$$\begin{aligned} B_{11,22} &= c^2 \\ B_{22,11} &= c^2 \\ B_{12,12} &= \frac{1}{2}\left(ab + c^2\right) \end{aligned} \tag{7}$$

The combination stated in the problem is then

$$2B_{12,12} - B_{11,22} - B_{22,11} = ab - c^2 \tag{8}$$

As pointed out by Zee on page 84, we can write 2 as a matrix equation in the form

$$z = \frac{1}{2}\vec{x}^T M \vec{x} \tag{9}$$

where

$$M \equiv \begin{bmatrix} a & c \\ c & b \end{bmatrix} \tag{10}$$

Note that, from 8

$$2B_{12,12} - B_{11,22} - B_{22,11} = \det M \tag{11}$$

If we rotate the coordinates by an angle θ, then the matrix M transforms according to

$$M' = R^{-1}MR \tag{12}$$

where R is the usual rotation matrix

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{13}$$

with inverse

$$R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{14}$$

If you multiply out 12 you get, for the off-diagonal elements of M':

$$M'_{12} = M'_{21} = 2c\cos^2\theta - (a-b)\sin\theta\cos\theta - c \tag{15}$$

To diagonalize M we would like these off-diagonal elements to be zero. We can use the trig identity

$$2\cos^2\theta - 1 = \cos^2\theta - \sin^2\theta \tag{16}$$

and then divide through by $\cos^2\theta$ to convert this to a quadratic equation for $\tan\theta$:

$$c\tan^2\theta + (a-b)\tan\theta - c = 0 \tag{17}$$

with solutions

$$\tan\theta = \frac{b-a\pm\sqrt{(a-b)^2+4c^2}}{2c} \tag{18}$$

The actual value of θ isn't important; what's important is that we *can* find a rotation angle that diagonalizes M (since the square root is always of a positive value, it's always real). From 12 and the fact that the determinant of a product of matrices is the product of the determinants, and that the determinant of an inverse matrix is the reciprocal of the original determinant, we see that

$$\det M' = \det M \tag{19}$$

and therefore that $ab-c^2$ is the product of the eigenvalues of M, which in turn is the product of the diagonal elements of M'. Diagonalizing M has the effect of writing the equation of the paraboloid in the form given in Zee:

$$z = \frac{1}{2}\mu u^2 + \frac{1}{2}\nu v^2 \tag{20}$$

where u and v are the coordinates obtained by rotating the (x,y) coordinates by the angle θ that diagonlizes M and μ and ν are the eigenvalues of M. The corresponding values for the metric as given in the problem statement are

$$\begin{aligned} g_{11} &= 1+\mu^2u^2 \\ g_{22} &= 1+\nu^2v^2 \\ g_{12} &= \mu\nu \end{aligned} \tag{21}$$

From this, we see from 5

$$\begin{aligned} B_{11,22} &= 0 \\ B_{22,11} &= 0 \\ B_{12,12} &= \frac{1}{2}g_{12} = \frac{1}{2}\mu\nu \end{aligned} \tag{22}$$

and therefore

$$2B_{12,12} - B_{11,22} - B_{22,11} = \mu\nu \tag{23}$$

I'm not sure that we can say that $\mu\nu$ is 'the' curvature, since if $\mu \neq \nu$ the paraboloid has different curvatures in the u and v directions, but the radius of curvature in the u direction is μ^{-1} and in the v direction it is ν^{-1}. If we are taking the product of the eigenvalues of M as the measure of the curvature that Zee is after, then from 11, we see that $2B_{12,12} - B_{11,22} - B_{22,11}$ does indeed provide the measure we're after.