DELTA FUNCTION: A COUPLE OF ALTERNATIVE DERIVATIONS

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We’ve been using the Dirac delta function frequently on this blog, and in some cases we’ve been using what looks like a dodgy formula in the form of an integral over a complex exponential:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$$ (1)

In fact, this formula can be made to look more credible by the following argument. Suppose we start with a finite integral:

$$d_K(x) \equiv \frac{1}{2\pi} \int_{-\frac{K}{2}}^{\frac{K}{2}} e^{ikx} dk$$ (2)

$$= \frac{1}{2\pi i x} e^{ixK} \bigg|_{-\frac{K}{2}}^{\frac{K}{2}}$$ (3)

$$= \frac{1}{\pi x} \sin \left( \frac{Kx}{2} \right)$$ (4)

As $$\lim_{x \to 0} \frac{\sin(ax)}{ax} = 1$$, this result has a peak of value $$K/2\pi$$ at $$x = 0$$ and oscillates on either side of the origin with decreasing amplitude as you get farther from the origin. Also, $$d_K(x) = 0$$ for the first time at $$x = \pm \frac{2\pi}{K}$$ on either side of the origin.

Using the standard integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$$ (5)

we find that

$$\int_{-\infty}^{\infty} d_K(x) dx = 1$$ (6)

Note that this result is independent of $$K$$, and remains true as $$K \to \infty$$. In this limit, the spike at $$x = 0$$ becomes infinitely large, and the width of the
spike becomes infinitesimal. Thus we can define the delta function as this limit:

\[ \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \, dk \]  

(7)

Another representation of \( \delta(x) \) is the formula

\[ \delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} \]  

(8)

For any nonzero value of \( \epsilon \), this function has a peak of height \( \frac{1}{\pi\epsilon} \) at \( x = 0 \), and its integral is

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\epsilon}{x^2 + \epsilon^2} \, dx = \frac{1}{\pi} \left[ \arctan \left( \frac{x}{\epsilon} \right) \right]_{-\infty}^{\infty} \]  

(9)

\[ = 1 \]  

(10)

As \( \epsilon \to 0 \), the peak height at \( x = 0 \) becomes infinite, and the peak width becomes zero, since for any \( x \neq 0 \) \( \delta(x) \to 0 \) as \( \epsilon \to 0 \), so it satisfies the requirements of a delta function.

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