The Gaussian integral can be used to define averages of powers of the integration variable \( x \). The average is defined as

\[
\langle x^{2n} \rangle \equiv \frac{\int_{-\infty}^{\infty} dx e^{-ax^2/2} x^{2n}}{\int_{-\infty}^{\infty} dx e^{-ax^2/2}}
\]  

We look only at even powers of \( x \) since the average of all odd powers is zero, as the integrand is odd. Rather than working out all the integrals, we can actually find \( \langle x^{2n} \rangle \) by differentiating the original Gaussian integral with respect to the parameter \( a \). We have

\[
G = \int_{-\infty}^{\infty} dx e^{-ax^2/2}
\]

\[
= \sqrt{\frac{2\pi}{a}}
\]

\[
-2 \frac{dG}{da} = \int_{-\infty}^{\infty} dx e^{-ax^2/2} x^2
\]

\[
= \sqrt{2\pi} \frac{1}{a^{3/2}}
\]

\[
(-2)^2 \frac{d^2 G}{da^2} = \int_{-\infty}^{\infty} dx e^{-ax^2/2} x^4
\]

\[
= \sqrt{2\pi} \frac{3}{a^{5/2}}
\]

\[
(-2)^n \frac{d^n G}{da^n} = \sqrt{2\pi} \frac{(2n-1)!!}{a^{(2n+1)/2}}
\]

where the double factorial is defined as

\[
(2n-1)!! \equiv (2n-1)(2n-3)\ldots3\times1
\]
Putting this result into \(2\) we get

\[
\langle x^{2n} \rangle = \frac{(2n - 1)!!}{a^n} \tag{11}
\]

We can also get this result from the variant Gaussian integral, obtained by completing the square in the exponent:

\[
\int_{-\infty}^{\infty} dx e^{-ax^2/2 + Jx} = \sqrt{\frac{2\pi}{a}} e^{J^2/2a} \tag{12}
\]

If we take the derivative of the LHS with respect to \(J\) we get

\[
\frac{d^{2n}}{dJ^{2n}} \int_{-\infty}^{\infty} dx e^{-ax^2/2 + Jx} = \int_{-\infty}^{\infty} dx e^{-ax^2/2 + Jx} x^{2n} \tag{13}
\]

Setting \(J = 0\) and comparing with \(2\) gives us

\[
\langle x^{2n} \rangle = \left. \frac{d^{2n}}{dJ^{2n}} e^{J^2/2a} \right|_{J=0} \tag{14}
\]

Although I could leave things here, I decided that it would be interesting to prove that the RHS actually does give \(\langle x^{2n} \rangle\). This proved to be a bit trickier than I expected, but the derivation is interesting so I’ll give it here.

We need to find a general expression for \(\frac{d^{2n}}{dJ^{2n}} e^{J^2/2a}\) before setting \(J = 0\). To help with this, we can write out the derivative explicitly for the first few values of \(n\). It’s actually easier to do the work for \(a = 1\); using dimensional analysis it’s easy enough to put \(a\) back in at the end. We find

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\frac{d^n}{dJ^n} e^{J^2/2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(e^{J^2/2} \left[ 1 + J^2 \right])</td>
</tr>
<tr>
<td>2</td>
<td>(e^{J^2/2} \left[ 3 + 6J^2 + J^4 \right])</td>
</tr>
<tr>
<td>3</td>
<td>(e^{J^2/2} \left[ 15 + 45J^2 + 15J^4 + J^6 \right])</td>
</tr>
<tr>
<td>4</td>
<td>(e^{J^2/2} \left[ 105 + 420J^2 + 210J^4 + 28J^6 + J^8 \right])</td>
</tr>
<tr>
<td>5</td>
<td>(e^{J^2/2} \left[ 945 + 4725J^2 + 3150J^4 + 630J^6 + 45J^8 + J^{10} \right])</td>
</tr>
</tbody>
</table>

We see that the constant term in each case is indeed equal to \((2n - 1)!!\). The coefficient of the second highest power of \(J\) is \((2n - 1) n\). Working backwards in each line, we find that the coefficient of the third highest power is \(\frac{1}{3} (2n - 1) (2n - 3) n (n - 1)\), of the fourth highest power is \(\frac{1}{3 \times 2} (2n - 1) (2n - 3) (2n - 5) n (n - 1) (n - 2)\) and so on. In general, the coefficient of \(J^{2n-2m}\) is

\[
\frac{n!}{m! (n-m)! (2n-2m-1)!!} \frac{(2n - 1)!!}{(2n-2m-1)!!} = \binom{n}{m} \frac{(2n - 1)!!}{(2n-2m-1)!!} \tag{15}
\]
Therefore we propose that

\[
\frac{d^{2n}}{dJ^{2n}} e^{J^2/2} = e^{J^2/2} \sum_{m=0}^{n} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} J^{2n-2m} \quad (16)
\]

We can prove this in general using induction. We’ve already established the anchor step, since this formula is true for \( n = 1 \ldots 5 \), so we can assume it for some value \( n \) and then work from there to prove it’s true for \( n + 1 \). That is, we want to show, starting from (16), that the following is true:

\[
\frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = e^{J^2/2} \sum_{m=0}^{n+1} \binom{n+1}{m} \frac{(2n+1)!!}{(2n-2m+1)!!} J^{2n-2m+2} \quad (17)
\]

We need to take the derivative of (16) twice. We get

\[
\frac{d^{2n+1}}{dJ^{2n+1}} e^{J^2/2} = e^{J^2/2} \sum_{m=0}^{n} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} \left[ J^{2n-2m+1} + (2n-2m) J^{2n-2m-1} \right] \quad (18)
\]

\[
\frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = e^{J^2/2} \sum_{m=0}^{n} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} \times \left[ J^{2n-2m+2} + (4n-4m+1) J^{2n-2m} + (2n-2m)(2n-2m-1) J^{2n-2m-2} \right] \quad (19)
\]

We’d like to put this in the form (17) so we can shift the summation index from \( m \to m-1 \) in the second term, and from \( m \to m-2 \) in the third term, thus allowing us to factor out \( J^{2n-2m+2} \) from all 3 terms. The limits on the sums will also change, so the second term now has limits of \( m = 1 \ldots n + 1 \) and the third term of \( m = 2 \ldots n + 2 \). We get (putting the exponential on the LHS for convenience):

\[
e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = \sum_{m=0}^{n} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} J^{2n-2m+2} + \\
\sum_{m=1}^{n+1} \binom{n}{m-1} \frac{(2n-1)!!}{(2n-2m+1)!!} (4n-4m+5) J^{2n-2m+2} + \\
\sum_{m=2}^{n+2} \binom{n}{m-2} \frac{(2n-1)!!}{(2n-2m+3)!!} (2n-2m+4) (2n-2m+3) J^{2n-2m+2} \quad (20)
\]

We can condense the sums by using the fact that the binomial coefficient \( \binom{p}{q} \) is zero if \( q < 0 \) or \( q > p \), so we can extend the lower limits on the second
and third sums to 0, and extend the upper limit on the first sum to \( n + 1 \). Also, in the third sum, the factor \((2n - 2m + 4)\) is zero when \( m = n + 2 \), so we can reduce the upper limit on the sum to \( n + 1 \). Therefore all three sums extend from \( m = 0 \) to \( n + 1 \) and we have (cancelling the common factor in the third sum as well):

\[
e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = \sum_{m=0}^{n+1} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} J^{2n-2m+2} + \\
\sum_{m=0}^{n+1} \binom{n}{m-1} \frac{(2n-1)!!}{(2n-2m+1)!!} (4n-4m+5) J^{2n-2m+2} + \\
\sum_{m=0}^{n+1} \binom{n}{m-2} \frac{(2n-1)!!}{(2n-2m+1)!!} (2n-2m+4) J^{2n-2m+2}
\]

(21)

To convert the binomial coefficients, we have

\[
\binom{n}{m-2} = \frac{m-1}{n-m+2} \binom{n}{m-1}
\]

(22)

This allows us to combine the second and third sums:

\[
e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = \sum_{m=0}^{n+1} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m-1)!!} J^{2n-2m+2} + \\
\sum_{m=0}^{n+1} \binom{n}{m-1} \frac{(2n-1)!!}{(2n-2m+1)!!} (4n-2m+3) J^{2n-2m+2}
\]

(23)

Also

\[
\binom{n}{m-1} = \frac{m}{n-m+1} \binom{n}{m}
\]

(24)

Using this, and multiplying the first sum by \( \frac{2n-2m+1}{2n-2m+1} \) we can combine it with the second sum to get

\[
e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = \sum_{m=0}^{n+1} \binom{n}{m} \frac{(2n-1)!!}{(2n-2m+1)!!} \left( 2n-2m+1 + \frac{m(4n-2m+3)}{n-m+1} \right) J^{2n-2m+2}
\]

(25)

Finally, we have
\[
\binom{n}{m} = \frac{n - m + 1}{n + 1} \binom{n + 1}{m} \tag{26}
\]

Plugging this in and simplifying, we get

\[
\left(2n - 2m + 1 + \frac{m(4n - 2m + 3)}{n - m + 1}\right) \binom{n}{m} = \left(2n - 2m + 1 + \frac{m(4n - 2m + 3)}{n - m + 1}\right) \frac{n - m + 1}{n + 1} \binom{n + 1}{m} \tag{27}
\]

\[
= (2n + 1) \binom{n + 1}{m} \tag{28}
\]

\[
e^{-J^2/2} \frac{d^{2n+2}}{dJ^{2n+2}} e^{J^2/2} = \sum_{m=0}^{n+1} \binom{n + 1}{m} \frac{(2n + 1)!!}{(2n - 2m + 1)!!} J^{2n-2m+2} \tag{29}
\]

QED.

To restore the factors of \(a\), we observe that \(J^2\) has the same dimensions as \(a\) (since an exponent must be dimensionless), so the derivative \(\frac{d^{2n}}{dJ^{2n}}\) has the dimensions of \(a^{-n}\). Therefore, the term involving \(J^{2n-2m}\) must be divided by \(a^{2n-m}\). Thus:

\[
\frac{d^{2n}}{dJ^{2n}} e^{J^2/2a} = e^{J^2/2a} \sum_{m=0}^{n} \binom{n}{m} \frac{(2n - 1)!!}{(2n - 2m - 1)!!} J^{2n-2m} \tag{30}
\]

When \(J = 0\), this reduces to the \(m = n\) term, which is (the double factorial \((-1)!! = 1\), at least according to Maple):

\[
\left. \frac{d^{2n}}{dJ^{2n}} e^{J^2/2a} \right|_{J=0} = \frac{(2n-1)!!}{a^n} \tag{31}
\]

which is the same as \(11\).

**Pingbacks**

Pingback: Gaussian integrals: averages over matrix components and the Wick contraction