NOETHER’S THEOREM

We’ve seen that for the Klein-Gordon field, there is a probability current that is conserved. It is defined by

\[ j_\mu = i \left( \phi^\dagger \partial_\mu \phi - \phi \partial_\mu \phi^\dagger \right) \]  (1)

and the conservation is expressed by its divergence being zero:

\[ \partial_\mu j_\mu = 0 \]  (2)

This is a special case of a more general principle known as Noether’s theorem. The idea is that if the Lagrangian density has a continuous symmetry, there is a corresponding current which is conserved.

First, what do we mean by a ‘continuous symmetry’? If we can vary the field \( \phi \) in some continuous manner (so that infinitesimal changes are possible) but in doing so, leave \( \mathcal{L} \) unchanged, then the Lagrangian has a continuous symmetry. In the more familiar language of 2-d geometry, for example, rotation of the coordinate system is a continuous operation, since we can rotate the coordinates by infinitesimal amounts. However, reflection of the coordinates about, say, the \( y \) axis, is not continuous since reflection is an all-or-nothing operation; there is no infinitesimal reflection.

Let’s suppose we have a general Lagrangian of the form \( \mathcal{L}(\phi_a, \partial_\mu \phi_a) \) where \( \mu \) indicates a spacetime coordinate and \( a \) enumerates the fields, as usual. Now suppose this Lagrangian is invariant when we change the fields by some infinitesimal amounts \( \delta \phi_a \). This means that \( \delta \mathcal{L} = 0 \) so we can say that

\[ \delta \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \phi_a} \delta \phi_a + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_a} \delta \partial_\mu \phi_a \]  (3)

Using the Euler-Lagrange equations for fields, which are...
\[
\frac{\delta L}{\delta \phi_a} - \partial_\mu \left( \frac{\delta L}{\delta \partial_\mu \phi_a} \right) = 0
\] (4)

we can write this as

\[
\delta L = \partial_\mu \left( \frac{\delta L}{\delta \partial_\mu \phi_a} \right) \delta \phi_a + \frac{\delta L}{\delta \partial_\mu \phi_a} \delta \partial_\mu \phi_a
\] (5)

Using the product rule, we can combine the two terms on the RHS to get

\[
\delta L = \partial_\mu \left( \frac{\delta L}{\delta \partial_\mu \phi_a} \right) = 0
\] (6)

That is, we’ve found a quantity (in parentheses) whose divergence is zero, so we can define this as a conserved current

\[
j^\mu \equiv \frac{\delta L}{\delta \partial_\mu \phi_a} \delta \phi_a
\] (7)

This is Noether’s theorem.

**Example 1.** As an example, we can use the Lagrangian for the Klein-Gordon equation

\[
L_0 = \partial_\mu \phi^\dagger \partial^\mu \phi - \mu^2 \phi^\dagger \phi
\] (8)

Treating \(\phi\) and \(\phi^\dagger\) as the two independent fields we note that since these two fields always occur as a product, the Lagrangian is unchanged if we replace

\[
\phi \rightarrow e^{i\theta} \phi; \quad \phi^\dagger \rightarrow e^{-i\theta} \phi^\dagger
\] (9)

This is a continuous symmetry, since the parameter \(\theta\) is a continuous variable. In infinitesimal form

\[
\begin{align*}
\phi & \rightarrow \phi + i\theta \phi \\
\delta \phi & = i\theta \phi \\
\phi^\dagger & \rightarrow \phi^\dagger - i\theta \phi^\dagger \\
\delta \phi^\dagger & = -i\theta \phi^\dagger
\end{align*}
\] (10-13)

We get

\[
\begin{align*}
\frac{\delta L}{\delta \partial_\mu \phi} & = \partial^\mu \phi^\dagger \\
\frac{\delta L}{\delta \partial_\mu \phi^\dagger} & = \partial^\mu \phi
\end{align*}
\] (14-15)
Therefore, from [7]

\[ j^\mu = i\theta\phi \partial^\mu \phi^\dagger - i\theta \phi^\dagger \partial^\mu \phi \]  
\[ = -i\theta \left( \phi^\dagger \partial_{\mu} \phi - \phi \partial_{\mu} \phi^\dagger \right) \]  

which apart from the \(-\theta\) (which drops out when taking the divergence and setting to zero) is the same current we had earlier in [1].

If you refer back to the derivation of the Euler-Lagrange equations, you’ll see that one of terms in the variation of the action was

\[ \int_{\Omega} \frac{\partial}{\partial q^\mu} \left[ \frac{\partial L}{\partial \phi_{a\mu}} \delta \phi_{a} \right] d^4 q \]  

where \( \Omega \) is the volume over which the integration is done. The argument made there was that since the integrand is a divergence, we can use Gauss’s theorem to convert this to a surface integral and since we’re holding the fields constant on the boundary, \( \delta \phi = 0 \) on the boundary, so this integral is zero. Using the same argument, if we add a divergence term, say \( \partial_{\mu} K^\mu \), to \( \delta L \), then the integral of this term over the volume is also zero, provided that \( K^\mu = 0 \) on the boundary. That is, we can replace [3] by

\[ \delta L = \partial_{\mu} \left( \frac{\delta L}{\delta \partial_{\mu} \phi_{a}} \delta \phi_{a} - K^\mu \right) = 0 \]  

so that the conserved current then becomes

\[ j^\mu \equiv \frac{\delta L}{\delta \partial_{\mu} \phi_{a}} \delta \phi_{a} - K^\mu \]  

[The derivation of this current in Peskin & Schroeder - their equation 2.12 - is a bit muddled. They refer to \( K^\mu \) as \( J^\mu \) and in the sentence before equation 2.12 they seem to state that \( \partial_{\mu} J^\mu = \partial_{\mu} \left( \frac{\delta L}{\delta \partial_{\mu} \phi_{a}} \delta \phi_{a} \right) \)  

and then state equation [20], which would of course imply that \( j^\mu = 0 \). The derivation in this post follows that given in Zee and is a lot clearer.]
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