RIEMANN TENSOR - COMMUTATOR OF RANK 2 TENSOR

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The covariant derivative of a contravariant vector is defined as

\[ \nabla_b V^a \equiv V^a_b \equiv \frac{\partial V^a}{\partial x^b} + V^c \Gamma^a_{cb} \]  

This is generalized to the covariant derivative of a higher-rank tensor by the formula

\[ T_{ab...}^{cd...} ; e = \partial_e T_{ab...}^{cd...} + T_{ab...}^{fe} \Gamma^a_{fe} + \ldots - T_{cd...}^{ab...} \Gamma^e_{de} - \ldots \]  

Ordinary partial derivatives, for a continuously differentiable function \( f(x^a) \), are commutative, that is

\[ \frac{\partial}{\partial x^b} \left( \frac{\partial f}{\partial x^a} \right) = \frac{\partial}{\partial x^a} \left( \frac{\partial f}{\partial x^b} \right) \]  

The covariant derivative, however, is not in general commutative, as we can verify by direct calculation. We want to find

\[ X_{b,c;d}^a - X_{b;d;c}^a \]  

which is known as the commutator of the tensor \( X^a_b \). For the first term, we get, using [2]

\[ X_{b,c;d}^a = \partial_d X_{b,c}^a + X_e^b \Gamma^a_{ec} - X_e^a \Gamma^c_{bd} - X_{b,c}^a \Gamma^e_{cd} \]  

\[ = \partial_d \left( \partial_c X^a_b + X_e^b \Gamma^a_{ec} - X_e^a \Gamma^c_{bd} \right) + \]  

\[ \Gamma^d_{ec} \left( \partial_c X^e_b + X_f^b \Gamma^e_{fc} - X_f^e \Gamma^d_{bc} \right) - \]  

\[ \Gamma^d_{bd} \left( \partial_c X^e_b + X_f^b \Gamma^a_{fc} - X_f^a \Gamma^d_{ec} \right) - \]  

\[ \Gamma^e_{cd} \left( \partial_c X^a_b + X_f^b \Gamma^a_{fc} - X_f^a \Gamma^e_{be} \right) \]  

The other term can be obtained by simply swapping the indices \( c \) and \( d \):
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\[ X^a_{bcd,e} = \partial_e X^a_{bcd} + X^e_{bd} \Gamma^a_{ec} - X^a_{ecd} \Gamma^e_{bc} - X^a_{bc} \Gamma^e_{dc} \]  

(10)

\[ = \partial_e (\partial_d X^a_b + X^e_b \Gamma^a_{ed} - X^a_e \Gamma^e_{bd}) + \]  

(11)

\[ \Gamma^a_{ec} \left( \partial_d X^e_b + X^f_b \Gamma^a_{fd} - X^e_f \Gamma^f_{bd} \right) - \]  

(12)

\[ \Gamma^e_{bc} \left( \partial_d X^a_e + X^f_e \Gamma^a_{fd} - X^a_f \Gamma^f_{ed} \right) - \]  

(13)

\[ \Gamma^e_{dc} \left( \partial_e X^a_b + X^f_b \Gamma^a_{fe} - X^a_f \Gamma^f_{be} \right) \]  

(14)

Now we need to take the difference. Assuming the ordinary partial derivatives commute and using the product rule, we get

\[ X^a_{bc:d} - X^a_{bcd:e} = X^b \left( \partial_d \Gamma^a_{ec} - \partial_e \Gamma^a_{ed} \right) - X^a \left( \partial_d \Gamma^e_{bc} - \partial_e \Gamma^e_{bd} \right) + \]  

(15)

\[ X^f_b \left( \Gamma^a_{ec} \Gamma^f_{de} - \Gamma^a_{ed} \Gamma^f_{ec} \right) - X^f_e \left( \Gamma^a_{ec} \Gamma^f_{bd} - \Gamma^a_{bd} \Gamma^f_{ec} \right) - \]  

(16)

\[ X^f_e \left( \Gamma^a_{bd} \Gamma^f_{ec} - \Gamma^a_{ec} \Gamma^f_{bd} \right) + X^a_f \left( \Gamma^a_{bd} \Gamma^e_{ec} - \Gamma^a_{ec} \Gamma^e_{bd} \right) - \]  

(17)

\[ \left( \partial_e X^a_b + X^f_b \Gamma^a_{fe} - X^a_f \Gamma^f_{be} \right) \left( \Gamma^e_{cd} - \Gamma^e_{dc} \right) \]  

(18)

We can now swap the indices \( e \) and \( f \) in the first term in the third line (since they are both dummy indices) to get

\[ X^f_e \left( \Gamma^a_{bd} \Gamma^f_{ec} - \Gamma^a_{ec} \Gamma^f_{bd} \right) = X^e_f \left( \Gamma^a_{bd} \Gamma^e_{ec} - \Gamma^a_{ec} \Gamma^e_{bd} \right) \]  

(19)

We can now see that this term cancels the last term on the second line. If we also assume that the affine connections are symmetric, so that

\[ \Gamma^e_{cd} = \Gamma^e_{dc} \]  

(20)

then the last line disappears and we are left with

\[ X^a_{bc:d} - X^a_{bcd:e} = X^e_b \left( \partial_d \Gamma^a_{ec} - \partial_e \Gamma^a_{ed} \right) - X^a_e \left( \partial_d \Gamma^e_{bc} - \partial_e \Gamma^e_{bd} \right) + \]  

(21)

\[ X^e_b \left( \Gamma^a_{fd} \Gamma^e_{fc} - \Gamma^a_{fc} \Gamma^e_{fd} \right) - X^a_e \left( \Gamma^f_{bd} \Gamma^e_{ec} - \Gamma^f_{ec} \Gamma^e_{bd} \right) \]  

(22)

\[ = X^e_b \left( \partial_d \Gamma^a_{ec} - \partial_e \Gamma^a_{ed} + \Gamma^a_{fd} \Gamma^e_{ec} - \Gamma^a_{fc} \Gamma^e_{fd} \right) - \]  

(23)

\[ X^a_e \left( \partial_d \Gamma^e_{bd} - \partial_e \Gamma^e_{bd} + \Gamma^f_{bd} \Gamma^e_{ec} - \Gamma^f_{ec} \Gamma^e_{bd} \right) \]  

(24)

where again we have swapped \( e \) and \( f \) in the second line.
The two terms in parentheses have the same form, and they are known as the Riemann tensor or curvature tensor, defined by

\[ R^a_{\, bcd} \equiv \partial_d \Gamma^a_{\, ec} - \partial_c \Gamma^a_{\, ed} + \Gamma^a_{\, fd} \Gamma^f_{\, ec} - \Gamma^a_{\, fc} \Gamma^f_{\, ed} \]  

(25)

In terms of the Riemann tensor, we get for the commutator:

\[ X^a_{\, b;cd} - X^a_{\, b;d;c} = X^e_{\, b} R^a_{\, edc} - X^a_{\, e} R^e_{\, bdc} \]  

(26)

This is actually the same result as given in d’Inverno’s problem 6.10, with \( c \) and \( d \) swapped around; I just took the original covariant derivatives in the opposite order to d’Inverno and can’t be bothered going through the whole derivation again to change it.

Pingbacks

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