RIEMANN TENSOR AND COVARIANT CONTRACTION

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Another exercise in fiddling with tensors.

The action of contracting a vector $X$ with the covariant derivative of a tensor (not sure if this operation has an official name, but I’ll call it *covariant contraction*) is (using the grad notation for covariant derivatives)

$$\nabla_X T^{ab\ldots}_{cd\ldots} \equiv X^e \nabla_e T^{ab\ldots}_{cd\ldots} \quad (1)$$

In the case of covariant contraction of a contravariant vector, we have

$$\nabla_X Z^a = X^e \nabla_e Z^a \quad (2)$$

If we combine two covariant contractions, we get

$$\nabla_X \nabla_Y Z^a = X^e \nabla_e (Y^c \nabla_c Z^a) = X^e \nabla_e Y^c \nabla_c Z^a + X^e Y^c \nabla_e \nabla_c Z^a \quad (3)$$

Swapping $X$ and $Y$ we get

$$\nabla_Y \nabla_X Z^a = Y^e \nabla_e X^c \nabla_c Z^a + Y^e X^c \nabla_e \nabla_c Z^a \quad (5)$$

We can swap the dummy indices $c$ and $e$ in the second term of this last equation to get

$$\nabla_Y \nabla_X Z^a = Y^e \nabla_e X^c \nabla_c Z^a + Y^c X^e \nabla_e \nabla_c Z^a \quad (6)$$

Taking the difference of these two derivatives we get

$$\nabla_X \nabla_Y Z^a - \nabla_Y \nabla_X Z^a = (X^e \nabla_e Y^c - Y^e \nabla_e X^c) \nabla_c Z^a + X^e Y^c (\nabla_e \nabla_c Z^a - \nabla_c \nabla_e Z^a) \quad (7)$$

By following through a similar derivation to that given for a rank-2 tensor we find that the factor in the last term can be written in terms of the Riemann tensor:

$$\nabla_e \nabla_c Z^a - \nabla_c \nabla_e Z^a = R^a_{bec} Z^b \quad (9)$$
Also, the factor in the first term is a Lie bracket

\[ X^c \nabla_e Y^c - Y^c \nabla_e X^c = [X,Y] \quad (10) \]

so we can write the first term as the covariant contraction with the vector given by the Lie bracket, or

\[ (X^c \nabla_e Y^c - Y^c \nabla_e X^c) \nabla_e Z^a = \nabla_{[X,Y]}Z^a \quad (11) \]

Combining all this we get the final result

\[ \nabla_X \nabla_Y Z^a - \nabla_Y \nabla_X Z^a - \nabla_{[X,Y]}Z^a = R^a_{\quad bce} X^e Y^c Z^b \quad (12) \]